

COMMUTATORS, C^k -CLASSIFICATION, AND SIMILARITY OF OPERATORS

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Abstract. We generalize the results of our recent paper, *The C^k -classification of certain operators in L_p* , II, to the abstract setting of a pair of operators satisfying the commutation relation $[M, N] = N^2$.

1. Introduction. The method we advanced in [11] for the solution of the C^k -classification and similarity problems for the operators $T_\zeta = M_x - \zeta J$ in $L_p(0, 1)$ ($1 < p < \infty$), where $(M_x f)(x) = xf(x)$ and $(Jf)(x) = \int_0^x f(t) dt$, turns out to work in much more general situations. The critical properties of M_x and J are the *commutation relation*

$$(1) \quad [M_x, J] = J^2$$

and, at a latter stage of the theory, the possibility of imbedding J in a holomorphic semigroup of operators J^ζ , possessing a boundary group on the imaginary axis. In this particular case, J^ζ is the Riemann-Liouville semigroup in $L^p(0, 1)$:

$$(2) \quad (J^\zeta f)(x) = \Gamma(\zeta)^{-1} \int_0^x (x-t)^{\zeta-1} f(t) dt, \quad \operatorname{Re} \zeta > 0.$$

The purpose of this paper is to study the implications of commutation relations of type (1), and more specifically, we shall generalize the results of [11] to our abstract setting.

We give now a more detailed description of our results. §2 contains three elementary lemmas, the first of which has very curious implications on the similarity of certain Banach algebra elements (cf. 7.1–7.3). We then study the commutation relation

$$(3) \quad [n, m] = an^2$$

in Banach algebras, where a commutes with both n and m . One reason for introducing the factor a is to get a unified treatment of $m + \zeta n$ and $m^* + \zeta n^*$ (when there is an involution $*$ in the algebra), for $[n, m] = n^2$ implies $[n^*, m^*] = -n^{*2}$, and both

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of these relations are of type (3). For other reasons, see §7 (in fact, we need only elements a which are scalar multiples of the identity). Of course, if the pair (n, m) satisfies (3), then the pair (an, m) satisfies (3), with $a=1$ (identity). The basic result of §3 is the "exponential-formula" (4) in Theorem 3.2, which generalizes Theorem 4 of [11]. As consequences, we obtain results concerning the spectrum of $m-kan$ (integral k) and its C^k -classification. For example, if the spectrum $\sigma(m)$ of m is real, then $\sigma(m-kan)=\sigma(m)$, and if m is also of class C^r for some $r \geq 0$, then $m-kan$ is of class $C^{r+|k|}$, and its $C^{r+|k|}$ -operational calculus is explicitly given (cf. Corollaries 3.9–3.10; the terminology, which is that of [9], is explained below).

In §4, we consider *regular semigroups of operators* on a Banach space. These are holomorphic semigroups $\{N(\zeta); \operatorname{Re} \zeta > 0\}$ of class (C_0) and finite type, the type being defined by

$$\nu = \sup_Q \log \|N(\zeta)\|$$

where Q is the rectangle $\{\zeta = \xi + i\eta; 0 < \xi \leq 1, |\eta| \leq 1\}$. For example, the Riemann-Liouville semigroup in $L_p(0, 1)$ ($1 < p < \infty$) is regular of type $\leq \pi/2$ (cf. [5]). It is known that a regular semigroup possesses a strongly continuous "boundary group" $\{N(i\eta); \eta \text{ real}\}$ of bounded operators, and the "extended" semigroup $\{N(\zeta); \operatorname{Re} \zeta \geq 0\}$ is of exponential type $\leq \nu$. If $\nu < \pi$, we show that for each ζ ($\operatorname{Re} \zeta > 0$), $N(\zeta)$ is a one-to-one operator with dense range. Defining $N(-\zeta) = N(\zeta)^{-1}$ ($\operatorname{Re} \zeta > 0$), we obtain a family of closed densely defined operators $\{N(\zeta); \zeta \text{ complex}\}$, bounded for $\operatorname{Re} \zeta \geq 0$, and satisfying the "semigroup relation" $N(\zeta_1 + \zeta_2) = N(\zeta_1)N(\zeta_2)$. Let $N = N(1)$, and suppose $A \neq 0$ and M are bounded operators such that A commutes with both M and N , and

$$(3') \quad [N, M] = AN^2.$$

We assume $\nu < \pi$. For the operators $T_\zeta = M + \zeta AN$, we prove the identities $N(\zeta)M = T_\zeta N(\zeta)$ and $MN(\zeta) = N(\zeta)T_{-\zeta}$ so that, in particular, T_ζ and T_α are similar if $\operatorname{Re} \zeta = \operatorname{Re} \alpha$.

In §5, we combine the ideas and results of §§3 and 4 to obtain generalizations of all the results of [11] to the present abstract setting. Let r, s denote nonnegative integers, and $\zeta = \xi + i\eta$ be any complex number. If M is real (i.e., $\sigma(M)$ real) of class C^r , then $\sigma(T_\zeta) = \sigma(M)$ and T_ζ is of class C^{r+s} in the strip $|\operatorname{Re} \zeta| \leq s$ (Theorem 5.2). The C^{r+s} -operational calculus for T_ζ is given in Theorem 5.5. More precise results are obtained when $\|e^{itM}\| = O(1)$ (which is the case for $M = M_x$). We then have a full generalization of the basic Theorem 6 of [11]. Write $c = \|AN\|$ ($c > 0$ of course) and $T_\zeta(t) = \exp(itT_\zeta)$, t real. Then there exist a constant H and a *positive* upper semicontinuous function $C(\xi)$ on \mathbb{R} such that

$$(4) \quad C(\xi)e^{-2\nu|\eta|} \leq (1 + c|t|)^{-|\zeta|} \|T_{\xi+i\eta}(t)\| \leq He^{2\nu|\eta|}$$

for all real ξ, η, t (Theorem 5.7).

From this we deduce a *precise* classification theorem which generalizes Theorem 1 in [11]:

THEOREM 5.8. *If M is real of class C , then T_ζ is of class C^s if and only if $|\operatorname{Re} \zeta| \leq s$.*

In §6, we turn to the similarity and spectrality problems for the operators T_ζ , and obtain generalizations of Theorems 2 and 3 of [11]. As before, $\|e^{itM}\| = O(1)$.

THEOREM 6.1. *T_ζ and T_α are not similar if $|\operatorname{Re} \zeta| \neq |\operatorname{Re} \alpha|$.*

THEOREM 6.2. *T_ζ is not spectral for $\operatorname{Re} \zeta \neq 0, 1, 2, \dots$*

Both results follow readily from (4). The method cannot decide the similarity question when $\operatorname{Re} \zeta = -\operatorname{Re} \alpha$ (a trivial case when $M = M_x$ and $N = J$, cf. [11, §3] or [6]), and the spectrality question when $\operatorname{Re} \zeta = 0, 1, 2, \dots$ (if $M = M_x$ and $N = J$, this case is easily settled by consideration of the point spectrum $\sigma_p(T_1)$; a generalization along these lines is given in Corollaries 6.4–6.5).

Various generalizations are studied in §7. In the Banach algebra context, Theorem 7.3 gives a similarity result (positive this time), which follows from Lemma 2.1 alone. Then, with A , $N(\cdot)$ and M as above, we obtain in particular that $M + \sum \alpha_i AN(\zeta_i)$ converges in the operator norm and is similar to M , whenever α_i , ζ_i are complex numbers such that $\operatorname{Re} \zeta_i \geq 1$, $\zeta_i \neq 1$ and $\sum \alpha_i (\zeta_i - 1)^{-1} N(\zeta_i - 1)$ converges. Thus, in all our results, T_α can be taken to mean

$$T_\alpha = M + \alpha AN + \sum \alpha_i AN(\zeta_i),$$

with α_i , ζ_i as above, and α complex arbitrary. Since α_i could be operators as well (if they commute with A , M , and N), we obtain as corollaries results concerning the operators $M + f(AN)$, where f is analytic at 0 (note that AN is quasi-nilpotent). For example, if $f(0)$ is real and M is real of class C , then $M + f(AN)$ is of class C^s if and only if $|\operatorname{Re} f'(0)| \leq s$. If f, g are analytic at 0, then $M + f(AN)$ and $M + g(AN)$ are similar if $f(0) = g(0)$ and $\operatorname{Re} f'(0) = \operatorname{Re} g'(0)$. Conversely, if $\|e^{itM}\| = O(1)$ and $M + f(AN) \sim M + g(AN)$, then $f(0) = g(0)$ and $|\operatorname{Re} f'(0)| = |\operatorname{Re} g'(0)|$. Here " $A \sim B$ " means " A is similar to B ". In particular, if $\|e^{itM}\| = O(1)$, then $M + f(AN) \sim M$ if and only if $f(0) = \operatorname{Re} f'(0) = 0$. We conclude the paper with some remarks on the reduction of more general commutation relations to the relation (3) (or (3')).

Note that if $A = \lambda I$ (λ complex), then our hypothesis on M , $N(\cdot)$ is "invariant under similarity", i.e. the pair $M', N'(\cdot)$ satisfies the same hypothesis if $M' = Q^{-1}MQ$ and $N'(\cdot) = Q^{-1}N(\cdot)Q$ (with the same nonsingular Q). For example, suppose φ is a strictly increasing map of $[0, 1]$ onto $[a, b]$ ($-\infty \leq a < b \leq \infty$), with $\varphi(0) = a$, $\varphi(1) = b$, such that $\varphi', 1/\varphi' \in L^\infty(0, 1)$. Then the substitution operator $S_\varphi: f(t) \rightarrow f(\varphi(x))$ is a nonsingular operator of $L_p(a, b)$ onto $L_p(0, 1)$ ($1 < p < \infty$), whose inverse is $S_\varphi^{-1} = S_\psi$, where ψ is the inverse function of φ . If we take $M = M_x$ and $N(\zeta) = J^\zeta$ in $L_p(0, 1)$, then $M' = M_\psi: f(t) \rightarrow \psi(t)f(t)$ and $N'(\cdot) = N_\psi(\cdot)$, where

$$(2') \quad [N_\psi(\zeta)f](t) = \Gamma(\zeta)^{-1} \int_a^t [\psi(t) - \psi(\tau)]^{\zeta-1} f(\tau) \psi'(\tau) d\tau,$$

$f \in L_p(a, b)$.

Thus our results are valid for the operators $T_\zeta = M' - \zeta N'$ ($= M_\psi - \zeta J M_{\psi'}$) on $L_p(a, b)$, which is of course trivial since they are similar to $M_x - \zeta J$. However, we may consider M' and $N'(\cdot)$ (given by (2')) with weaker assumptions on ψ (so that $M' - \zeta N'$ is not necessarily similar to $M_x - \zeta J$), say ψ' exists a.e. and belongs to $L^\infty(a, b)$. We can verify that $N'(\cdot)$ is a regular semigroup of type $\leq \pi/2$ and $[M_\psi, N_\psi] = N_\psi^2$. For ψ real, our results are valid for the pair M_ψ, N_ψ . For example, if $\varphi(x) = e^x$ ($0 \leq x \leq 1$), $N_\psi(\cdot)$ is the "Hadamard semigroup" [4, p. 672].

$$[N_\psi(\zeta)f](t) = \Gamma(\zeta)^{-1} \int_1^t [\log(t/\tau)]^{\zeta-1} f(\tau) \frac{d\tau}{\tau},$$

$f \in L_p(1, e)$. In particular, our results are valid for the operators

$$T_\zeta: f(t) \rightarrow (\log t)f(t) - \zeta \int_1^t f(\tau) \frac{d\tau}{\tau}$$

on $L_p(1, e)$.

A completely different method for settling the similarity question (only) for the family $M_\psi - \zeta J M_{\psi'}$ was advanced by G. Kalisch in [6]. It was Kalisch who asked me at the Madison Mathematical Meeting (Summer, 1968), whether my method, as presented in [11], would apply to abstract semigroups satisfying a commutation relation (3'); I wish to thank him for bringing this problem to my attention.

NOTATIONS. \mathbf{R}, \mathbf{C} : the real and complex fields respectively.

I : the identity operator (on a Banach space).

If x belongs to a Banach algebra A with identity, $\sigma(x)$, $\sigma_p(x)$, $\rho(x)$ and $R(\lambda; x)$ denote (respectively) the spectrum, the point spectrum, the resolvent set, and the resolvent of x .

$C^r[a, b]$: the Banach algebra of all complex functions of class C^r ($r \geq 0$) on $[a, b] \subset \mathbf{R}$, with the norm

$$\|\varphi\|_r = \sum_{j=0}^r \sup_{[a,b]} |\varphi^{(j)}|/j!.$$

Suppose $\sigma(x) \subset [a, b]$. The element $x \in A$ is of class C^r (or $C^r[a, b]$) if there exists a continuous homomorphism $\varphi \rightarrow \varphi(x)$ of $C^r[a, b]$ into A such that $1 \rightarrow 1$ and $\varphi(t) \equiv t \rightarrow x$. This homomorphism is unique when it exists, and is called the C^r -operational calculus for x (cf. [9]). Clearly, x is of class C^r iff there exists a constant K such that $\|\varphi(x)\| \leq K \|\varphi\|_r$ for all *polynomials* φ .

The paper is virtually self-contained; the only essential references are to a couple of theorems in Hille-Phillips [4].

2. Three elementary lemmas. If f is a polynomial over a field K , we shall denote by f' the formal derivative of f . If A is an (associative) algebra with identity 1 over K , then f operates on A in the usual way: $f(x) = \sum \lambda_k x^k$ for $x \in A$ and $f(\xi) = \sum \lambda_k \xi^k$.

LEMMA 2.1 (THE CHAIN RULE). *Let D be a derivation on A , and let $n \in A$ be such that Dn commutes with n . Then $Df(n) = f'(n)Dn$ for every polynomial f over K .*

If A is a Banach algebra over \mathbb{C} , and D is a continuous derivation on A , then the chain rule is valid for all complex functions f analytic in a neighborhood of $\sigma(n)$.

Proof. Since $D1=0$, the first part of the lemma follows from the identity

$$(1) \quad Dn^k = kn^{k-1}Dn, \quad k = 1, 2, \dots$$

We prove (1) by induction on k . This is trivial for $k=1$. Assuming (1) for some k , we then have

$$\begin{aligned} Dn^{k+1} &= D(n^k n) = (Dn^k)n + n^k Dn \\ &= kn^{k-1}(Dn)n + n^k Dn = (k+1)n^k Dn \end{aligned}$$

since Dn commutes with n .

If $a \in A$ is invertible, $0 = D1 = D(aa^{-1}) = (Da)a^{-1} + aDa^{-1}$, and thus $Da^{-1} = -a^{-1}(Da)a^{-1}$. Suppose A is a Banach algebra over \mathbb{C} , and $\lambda \in \rho(n)$. Then

$$D(\lambda 1 - n)^{-1} = (\lambda 1 - n)^{-1} Dn (\lambda 1 - n)^{-1} = (\lambda 1 - n)^{-2} Dn.$$

Therefore, if f is a complex function analytic in a neighborhood of $\sigma(n)$, then for a suitable contour C , the continuity of D implies that

$$\begin{aligned} 2\pi i Df(n) &= \int_C f(\lambda) D(\lambda 1 - n)^{-1} d\lambda \\ &= \int_C f(\lambda) (\lambda 1 - n)^{-2} d\lambda Dn = 2\pi i f'(n) Dn. \end{aligned}$$

Note that f can be A -valued (cf. [7]) provided D commutes with multiplication by $f(\lambda)$ for each λ . For some curious consequences of this lemma, see §7.

For A as before and $n \in A$, let D_n be the inner derivation generated by n : $D_n x = [n, x] = nx - xn$ ($x \in A$). The following "second chain rule" is valid in A .

LEMMA 2.2. Let a, m, n be elements of A such that

- (i) a commutes with m and n ; and
- (ii) $D_n m = an^2$.

Then

- (1) $D_n f(m) = anf'(m)n$ for every polynomial f over K .

If A is a complex Banach algebra, (1) is valid for every complex function f analytic in a neighborhood of the disc $\{\lambda; |\lambda| \leq \|m\|\}$; if $\sigma(m)$ has a connected complement, (1) is valid for every f analytic in a neighborhood of $\sigma(m)$.

Proof. We prove by induction on k that

- (2) $D_n m^k = kanm^{k-1}n$, $k = 1, 2, \dots$

This obviously implies the first part of the lemma. For $k=1$, (2) reduces to the hypothesis (ii). Suppose (2) is valid for some $k \geq 1$; then by (i) and (ii)

$$D_n m^{k+1} = (D_n m^k)m + m^k D_n m = kanm^{k-1}nm + am^k n^2.$$

By (ii) and (2) for k , we have

$$nm = mn + an^2 \quad \text{and} \quad m^k n = nm^k - kanm^{k-1}n.$$

Therefore

$$\begin{aligned} D_n m^{k+1} &= kanm^{k-1}(mn + an^2) + a(nm^k - kanm^{k-1}n)n \\ &= (k+1)anm^k n, \quad \text{as wanted.} \end{aligned}$$

The second part of the lemma follows from the first by Runge's theorem and the continuity of the analytic operational calculus.

Let $(G, +)$ be an abelian group, and let (H, \cdot) be a semigroup with identity 1. A map $f: G \rightarrow H$ is called a *homomorphism* if $f(0)=1$ and $f(x+y)=f(x)f(y)$ for all $x, y \in G$. A map $g: G \rightarrow H$ is *abelian* if its range is an abelian subset of H . The product of two maps f, g of G into H is defined pointwise: $(fg)(x)=f(x)g(x)$.

LEMMA 2.3. *Let G be an abelian group, and H a semigroup with identity. Let $f, g: G \rightarrow H$ be such that f and fg are homomorphisms and g is abelian. Then fg^r is a homomorphism for all $r=0, \pm 1, \pm 2, \dots$*

Proof. Since f and $h=fg$ are homomorphisms, $f(x)$ and $h(x)$ are invertible in H , and therefore $g(x)=f(x)^{-1}h(x)$ is invertible for each $x \in G$. Since g is abelian, $g(x)^{-1}$ commutes with $g(y)^{-1}$ and with $g(y)$ for all $x, y \in G$. Let $k=fg^{-1}$. Then $k(0)=1$ and for all $x, y \in G$,

$$\begin{aligned} k(x+y) &= f(x)f(y)g(x+y)^{-1} = f(x)f(y)g(x+y)^{-1}g(y)g(y)^{-1} \\ &= f(x)f(y)g(y)g(x+y)^{-1}g(y)^{-1} \\ &= f(x)h(y)h(x+y)^{-1}f(x)f(y)g(y)^{-1} \\ &= f(x)h(x)^{-1}f(x)k(y) \\ &= f(x)g(x)^{-1}k(y) = k(x)k(y), \end{aligned}$$

i.e. k is a homomorphism.

Thus, the pair (f, g^{-1}) satisfies the same hypothesis as the pair (f, g) , and consequently, it suffices to prove the lemma for *positive* integers r . This is done by induction on r . For $r=1$, there is nothing to prove. Assume fg^r is a homomorphism for some $r \geq 1$. Then, for all $x, y \in G$,

$$\begin{aligned} f(x+y)g(x+y)^{r+1} &= f(x+y)g(x+y)^r g(x+y) \\ &= f(x)g(x)^r f(y)g(y)^r g(x+y) \\ &= f(x)g(x)^r [f(y)g(x+y)]g(y)^r \end{aligned}$$

since g is abelian.

Since f and $h=fg$ are homomorphisms,

$$f(x)f(y)g(x+y) = f(x+y)g(x+y) = h(x)h(y) = f(x)g(x)f(y)g(y),$$

i.e.

$$(1) \quad f(y)g(x+y) = g(x)f(y)g(y).$$

Thus

$$\begin{aligned} f(x+y)g(x+y)^{r+1} &= f(x)g(x)^r[g(x)f(y)g(y)]g(y)^r \\ &= f(x)g(x)^{r+1}f(y)g(y)^{r+1} \end{aligned}$$

for all $x, y \in G$.

Since $g(0)=1$, fg^{r+1} is a homomorphism, and the proof is complete.

REMARK. Taking $y=-x$ in (1), we see that $g(x)^{-1}=f(x)^{-1}g(-x)f(x)$, i.e. $g(x)^{-1}$ is similar to $g(-x)$ for all $x \in G$.

3. An exponential formula. Throughout this section, A is a complex Banach algebra with identity 1, and a, m, n are fixed elements of A such that

- (i) a commutes with m and n ; and
- (ii) $[n, m]=an^2$ (except in the second part of Theorem 3.2).

LEMMA 3.1. an is quasi-nilpotent, and for all complex λ and integers k ,

$$(1) \quad e^{\lambda(m-kan)} = e^{\lambda m}(1-\lambda an)^k = (1+\lambda an)^{-k}e^{\lambda m}.$$

Proof. Let $n_1=an$. Then $[n_1, m]=a[n, m]=a^2n^2=n_1^2$. This shows that we may assume without loss of generality that $a=1$.

By Lemma 2.1, $[n^k, m]=kn^{k+1}$, hence $\|n^{k+1}\| \leq (2\|m\|/k)\|n^k\|$, and by induction

$$(2) \quad \|n^p\| \leq \frac{(2\|m\|)^{p-1}}{(p-1)!} \|n\|, \quad p = 1, 2, \dots$$

Thus $\|n^p\|^{1/p} \rightarrow 0$ and n is quasi-nilpotent. Note that since $[n, m]=n^2$ commutes with n , we could appeal to Theorem 1.3.1 in [12] in order to obtain the quasi-nilpotency of n^2 , hence of n .

Since $\sigma(n)=\{0\}$, $(1+\lambda n)^k$ is well defined for all complex λ and all integers k . It suffices to prove the first relation in (1), because then

$$e^{\lambda(m-kan)} = [e^{-\lambda(m-kan)}]^{-1} = [e^{-\lambda m}(1+\lambda n)^k]^{-1} = (1+\lambda n)^{-k}e^{\lambda m}.$$

All the functions of λ appearing in (1) are analytic in \mathbb{C} and have the same derivative at 0. Consequently, the lemma will be proved if we show that $e^{\lambda m}(1-\lambda n)^k$ is a group. Now, by Lemma 2.3 with $G=\mathbb{C}$, $H=A$, $f(\lambda)=e^{\lambda m}$ and $g(\lambda)=1-\lambda n$, it suffices to prove that $e^{\lambda m}(1-\lambda n)$ is a group. By Lemma 2.2 with $a=1$ and $f(\zeta)=e^{\mu \zeta}$ ($\mu \in \mathbb{C}$ fixed), we have

$$ne^{\mu m} - e^{\mu m}n = \mu ne^{\mu m}n \quad (\mu \in \mathbb{C}).$$

Thus, for all $\lambda, \mu \in \mathbb{C}$,

$$\begin{aligned} (1-\lambda n)e^{\mu m}(1-\mu n) &= e^{\mu m}(1-\mu n) - \lambda(ne^{\mu m} - \mu ne^{\mu m}n) \\ &= e^{\mu m}(1-\mu n) - \lambda e^{\mu m}n = e^{\mu m}(1-(\lambda+\mu)n). \end{aligned}$$

Multiplying both sides by $e^{\lambda m}$, we obtain the group property for $e^{\lambda m}(1-\lambda n)$, and the proof is complete.

We now extend Lemma 3.1 to complex k . Note first that since an is quasi-nilpotent, $(1 - \lambda an)^\mu$ is well defined for all $\lambda, \mu \in \mathbb{C}$ by means of the analytic operational calculus, and for each fixed λ , it is a holomorphic group on \mathbb{C} .

$$(3) \quad (1 - \lambda an)^\mu = \frac{1}{2\pi i} \int_{\Gamma} \zeta^\mu [\zeta - 1 + \lambda an]^{-1} d\zeta, \quad \lambda, \mu \in \mathbb{C},$$

where $\Gamma = \{\zeta \in \mathbb{C}; |\zeta - 1| = \frac{1}{2}\}$, to fix the ideas.

THEOREM 3.2. *Let A be a complex Banach algebra with identity 1, and let a, m, n be elements of A such that a commutes with m and n . Suppose*

$$(ii) \quad [n, m] = an^2.$$

Then, for all complex λ, μ ,

$$(4) \quad e^{\lambda(m - \mu an)} = e^{\lambda m} (1 - \lambda an)^\mu = (1 + \lambda an)^{-\mu} e^{\lambda m}.$$

Conversely, if a is invertible and if, for some $\mu \neq 0$, one of the relations (4) holds (for all $\lambda \in \mathbb{C}$), then (ii) holds.

Proof. It follows from the definition (3) that $d(1 - \lambda an)^\mu/d\lambda = -\mu an(1 - \lambda an)^{\mu-1}$; therefore, if one of the relations (4) is valid for some $\mu \neq 0$, a straightforward comparison of the second derivatives with respect to λ at $\lambda=0$ gives $a[n, m] = a^2 n^2$, i.e. (ii) holds if a is invertible.

We prove now the first part of the theorem. As in Lemma 3.1, we may assume $a=1$. Since (4) is trivial for $n=0$, we assume $n \neq 0$, and let $\delta = \pi/(2\|n\|)$ and

$$K = \max \{ \|(\zeta - 1 + \lambda n)^{-1}\|; \zeta \in \Gamma, |\lambda| \leq \delta \}.$$

For $\zeta = re^{i\theta}$ varying on Γ , we have $1/2 \leq r \leq 3/2$ (so surely $|\log r| < 1$) and $|\theta| \leq \pi/6$. Therefore, if $\mu = \beta + i\gamma$,

$$\begin{aligned} \log |\zeta^\mu| &= \log r^\beta e^{-\gamma\theta} = \beta \log r - \gamma\theta \\ &\leq |\beta| |\log r| + |\theta| |\gamma| \leq |\mu| (1 + \pi/6) \leq \pi |\mu|/2. \end{aligned}$$

Thus, by (3),

$$(5) \quad \|(1 - \lambda n)^\mu\| \leq K e^{\pi |\mu|/2} \quad (|\lambda| \leq \delta).$$

Also

$$\begin{aligned} \|e^{-\lambda m} e^{\lambda(m - \mu n)}\| &\leq e^{|\lambda| \|m\|} e^{|\lambda| \|m - \mu n\|} \\ &\leq e^{2|\lambda| \|m\|} e^{|\lambda| |\mu| \|n\|} \leq e^{\pi \|m\|/\|n\|} e^{\pi |\mu|/2} \end{aligned}$$

for $|\lambda| \leq \delta$.

For $\lambda \in \mathbb{C}$ fixed, consider the entire function of μ ,

$$F_\lambda(\mu) = e^{-\lambda m} e^{\lambda(m - \mu n)} - (1 - \lambda n)^\mu.$$

If $|\lambda| \leq \delta$, $F_\lambda(\cdot)$ is of exponential type $\leq \pi/2$ and $F_\lambda(k) = 0$ for all integers k by Lemma 3.1. Therefore $F_\lambda(\mu) = 0$ for all $\mu \in \mathbb{C}$ and $|\lambda| \leq \delta$. Since $F_\lambda(\mu)$ is also an entire function of λ , it follows that $F_\lambda(\mu) = 0$ for all $\lambda, \mu \in \mathbb{C}$, and the first relation in (4) is proved. The second relation follows from the first as in Lemma 3.1.

For $\lambda \in C$ fixed, consider the binomial series

$$B_\lambda(\mu) = \sum_{j=0}^{\infty} (-1)^j \binom{\mu}{j} \lambda^j a^j n^j \quad (\mu \in C),$$

where $\binom{\mu}{0} = 1$ and $\binom{\mu}{j} = \mu(\mu-1) \cdots (\mu-j+1)/j!$ for positive integers j . If μ is a positive integer, $\binom{\mu}{j} = 0$ for $j > \mu$ and $B_\lambda(\mu)$ is the binomial expansion of $(1 - \lambda an)^\mu$. Since $|\binom{\mu}{j}| \leq (1 + |\mu|)^j$, it follows from (2) that $B_\lambda(\mu)$ converges absolutely for all $\lambda, \mu \in C$. As a binomial series in μ , it is therefore of exponential type $\leq \pi/2$ for $\operatorname{Re} \mu > 0$ (λ fixed). By (5), $(1 - \lambda an)^\mu$ is also of exponential type $\leq \pi/2$ as a function of μ ($\mu \in C, \lambda$ fixed with $|\lambda| \leq \delta$). Since $B_\lambda(\mu) = (1 - \lambda an)^\mu$ for positive integers μ , the same relation is valid for all $\mu, \lambda \in C$ such that $\operatorname{Re} \mu > 0$ and $|\lambda| \leq \delta$, and hence for all $\mu, \lambda \in C$ since both functions are entire in λ and μ . We proved

COROLLARY 3.3.

$$\begin{aligned} e^{\lambda(m - \mu an)} &= e^{\lambda m} \sum_{j=0}^{\infty} (-1)^j \binom{\mu}{j} \lambda^j a^j n^j \\ &= \sum_{j=0}^{\infty} \binom{-\mu}{j} \lambda^j a^j n^j e^{\lambda m} \end{aligned}$$

for all $\lambda, \mu \in C$, and the series converge absolutely.

Expanding the exponentials in powers of λ , multiplying the series on the right and comparing the coefficients of λ^k , we obtain

COROLLARY 3.4. For all complex μ and nonnegative integers k ,

$$\begin{aligned} (m - \mu an)^k &= \sum_{j=0}^k (-1)^j \binom{\mu}{j} \frac{k!}{(k-j)!} m^{k-j} a^j n^j \\ &= \sum_{j=0}^k \binom{-\mu}{j} \frac{k!}{(k-j)!} a^j n^j m^{k-j}. \end{aligned}$$

We note at this point that since all functions of μ appearing in Corollary 3.4 are polynomials, the corollary is equivalent to its special case with μ a positive integer. This case can be proved directly by induction on k , using Lemma 2.2, and this leads to an alternative proof of Theorem 3.2. This approach shows also that Corollary 3.4 is valid in the general setting of Lemma 2.2 (first part!). We preferred the above method because the inductive proof is quite tedious.

COROLLARY 3.5. For all complex μ and all polynomials φ ,

$$\begin{aligned} \varphi(m - \mu an) &= \sum_{j=0}^{\infty} (-1)^j \binom{\mu}{j} \varphi^{(j)}(m) a^j n^j \\ &= \sum_{j=0}^{\infty} \binom{-\mu}{j} a^j n^j \varphi^{(j)}(m) \end{aligned}$$

(the sums are of course finite since $\varphi^{(j)} = 0$ for large j).

Proof. For $\varphi(t) = t^k$ ($k=0, 1, 2, \dots$), this is just Corollary 3.4. The general case follows then from the linearity of all the expressions involved when considered as functions of φ .

NOTATION. $\sigma^*(m) = \{\zeta \in C; \text{dist}(\zeta, \sigma(m)) \leq 2\|m\|\}$.

THEOREM 3.6. *If $\sigma^*(m)$ has a connected complement, then for all complex μ ,*

- (a) $\sigma(m - \mu an) \subset \sigma^*(m)$;
- (b) *the identities of Corollary 3.5 are valid for every complex function φ holomorphic in a neighborhood of $\sigma^*(m)$, and the series converge absolutely;*
- (c) *in particular, for $\zeta \notin \sigma^*(m)$, the following identities hold:*

$$\begin{aligned} R(\zeta; m - \mu an) &= \sum_{j=0}^{\infty} (-1)^j \binom{\mu}{j} j! R(\zeta; m)^{j+1} a^j n^j \\ &= \sum_{j=0}^{\infty} \binom{-\mu}{j} j! a^j n^j R(\zeta; m)^{j+1}. \end{aligned}$$

For each fixed μ , the series converge absolutely and uniformly in ζ on every compact subset of $C \setminus \sigma^(m)$.*

Proof. Fix $\mu \in C$, $\zeta \in C \setminus \sigma^*(m)$ and $0 < h < 1$ such that $2\|m\|/h \text{dist}(\zeta; \sigma(m)) < 1$. Since $\sum_j \binom{\mu}{j} h^j$ converges absolutely, there exists a constant K such that $|\binom{\mu}{j}| \leq Kh^{-j}$ for all $j=1, 2, \dots$. By (2) (with n replaced by an), the j th term of the series in (c) has norm smaller than or equal to $K\|an\|jh^{-j}(2\|m\|)^{j-1}\|R(\zeta; m)^{j+1}\|$, and therefore the lim sup of its j th root is $\leq 2\|m\|/h \text{dist}(\zeta; \sigma(m)) < 1$ (cf. [7, p. 530]). Thus the series converge absolutely for $\zeta \in C \setminus \sigma^*(m)$, and if $Q \subset C \setminus \sigma^*(m)$ is compact, similar estimates show the uniform convergence of the series on Q . In particular, the series in (c) define functions holomorphic for $\zeta \in C \setminus \sigma^*(m)$, and since $C \setminus \sigma^*(m)$ is connected, it suffices to prove the identities (c) for large $|\zeta|$ (indeed, if $F(\zeta)$ denotes any one of the series in (c), the relation

$$[\zeta - (m - \mu an)]F(\zeta) = F(\zeta)[\zeta - (m - \mu an)] = 1$$

is valid for all $\zeta \in C \setminus \sigma^*(m)$ as soon as it is valid for large $|\zeta|$). For such ζ , we have by Corollary 3.4,

$$\begin{aligned} R(\zeta; m - \mu an) &= \sum_{k=0}^{\infty} \zeta^{-k-1} (m - \mu an)^k \\ &= \sum_{k=0}^{\infty} \zeta^{-k-1} \sum_{j=0}^k (-1)^j \binom{\mu}{j} \frac{k!}{(k-j)!} m^{k-j} a^j n^j \\ &= \sum_{j=0}^{\infty} (-1)^j \binom{\mu}{j} \sum_{k=j}^{\infty} \frac{k!}{(k-j)!} \zeta^{-k-1} m^{k-j} a^j n^j \\ &= \sum_{j=0}^{\infty} (-1)^j \binom{\mu}{j} (-1)^j R(\zeta; m)^{j+1} a^j n^j \\ &= \sum_{j=0}^{\infty} (-1)^j \binom{\mu}{j} j! R(\zeta; m)^{j+1} a^j n^j. \end{aligned}$$

The change of the order of summation is justified as soon as the series

$$\sum_{k=0}^{\infty} |\zeta|^{-k-1} \sum_{j=0}^k \left| \binom{\mu}{j} \right| \frac{k!}{(k-j)!} \|m^{k-j}\| \|(an)^j\|$$

converges. By (2), this is surely the case if $|\zeta| > 3\|m\|$. The same estimates justify also the first equation above for $|\zeta| > 3\|m\|$ (cf. Corollary 3.4). This proves the first identity in (c) and (a). Similar computations derive the second identity in (c) from the second relation in Corollary 3.4. Finally, (b) is derived from (c) by contour integration term-by-term (note that, with the usual choice of Γ ,

$$\frac{1}{2\pi i} \int_{\Gamma} \varphi(\zeta) j! R(\zeta; m)^{j+1} d\zeta = \varphi^{(j)}(m), \quad j=0, 1, 2, \dots).$$

COROLLARY 3.7. *If $\sigma(m)$ has a connected complement, then for all integers k ,*

(a) $\sigma(m-kan) \subset \sigma(m)$;

(b) *for all complex functions φ holomorphic in a neighborhood of $\sigma(m)$,*

$$\begin{aligned} \varphi(m-kan) &= \sum_{j=0}^k (-1)^j \binom{k}{j} \varphi^{(j)}(m) a^j n^j, & k \geq 0, \\ &= \sum_{j=0}^{-k} \binom{-k}{j} a^j n^j \varphi^{(j)}(m), & k < 0; \end{aligned}$$

(c) *in particular, for all $\zeta \notin \sigma(m)$,*

$$\begin{aligned} R(\zeta; m-kan) &= \sum_{j=0}^k (-1)^j \binom{k}{j} j! R(\zeta; m)^{j+1} a^j n^j, & k \geq 0, \\ &= \sum_{j=0}^{-k} \binom{-k}{j} j! a^j n^j R(\zeta; m)^{j+1}, & k < 0. \end{aligned}$$

Proof. (c) is a special case of Theorem 3.6(c) (for large $|\zeta|$), since $\binom{\mu}{j} = 0$ for $j > \mu$ if μ is a nonnegative integer. Since the right side is holomorphic in the connected set $\rho(m)$, an argument made in the preceding proof shows that (c) is valid throughout $\rho(m)$. In particular, (a) holds, and finally, (c) implies (b) by contour integration.

COROLLARY 3.8. *Let k be an integer. Then $\sigma(m)$ and $\sigma(m-kan)$ coincide if both have connected complements.*

Proof. Fix k , and let $m' = m - kan$. Then a commutes with m' and $[n, m'] = [n, m] = an^2$. Thus the triple (a, m', n) satisfies the hypothesis required of (a, m, n) . Therefore, by Corollary 3.7(a) with k, m replaced by $-k, m'$, we have $\sigma(m) = \sigma(m' + kan) \subset \sigma(m')$. Since $\sigma(m') \subset \sigma(m)$ by Corollary 3.7(a), the proof is complete.

COROLLARY 3.9. *If $\sigma(m)$ is real, then $\sigma(m-kan) = \sigma(m)$ for all integers k .*

Proof. By Corollary 3.7(a), $\sigma(m-kan)$ is also real, and we may then apply Corollary 3.8.

(Of course, the same result holds with the real line replaced by a suitable Jordan curve.)

COROLLARY 3.10. *Let m be real of class C^r for some $r \geq 0$. Then, for any integer k , $m - kan$ is real of class $C^{r+|k|}$. Moreover, if $\varphi \rightarrow \varphi(m)$ denotes the C^r -operational calculus for m , then the $C^{r+|k|}$ -operational calculus for $m - kan$ is given by $\varphi \rightarrow \varphi(m - kan)$, with $\varphi(m - kan)$ defined by (b) in Corollary 3.7.*

Proof. Fix real numbers $\alpha < \beta$ such that $(\alpha, \beta) \supset \sigma(m)$. The map $\varphi \rightarrow \varphi(m - kan)$ defined by (b) in Corollary 3.7 is a well-defined continuous linear mapping of $C^{r+|k|}[\alpha, \beta]$ into A , and it coincides by Corollary 3.7(b) with the usual operational calculus when restricted to polynomials φ . In particular, the map is multiplicative on polynomials, hence on $C^{r+|k|}[\alpha, \beta]$, since the polynomials form a dense subclass.

Note that Corollaries 3.7–3.10 depend on Lemma 3.1 rather than Theorem 3.2, so that more elementary proofs are possible.

4. Regular semigroups of operators. Throughout the following sections, X is a complex Banach space, $B(X)$ is the Banach algebra of all (bounded linear) operators on X , C^+ is the right half-plane $\{\zeta \in C; \operatorname{Re} \zeta > 0\}$, and $N(\cdot): C^+ \rightarrow B(X)$ is a semigroup of operators. The *type* of $N(\cdot)$ is

$$\nu = \sup_Q \log \|N(\zeta)\|$$

where $Q = \{\zeta = \xi + i\eta; 0 < \xi \leq 1, |\eta| \leq 1\}$. $N(\cdot)$ is of class (C_0) if for each $x \in X$, $N(\xi)x \rightarrow x$ strongly as $\xi \rightarrow 0+$ (cf. [4, p. 321]). We shall write $N = N(1)$.

DEFINITION 4.1. $N(\cdot)$ is *regular* if it is holomorphic of class (C_0) and finite type.

For example, the Riemann-Liouville semigroup J^t in $L_p(0, 1)$ ($1 < p < \infty$) is regular (cf. [4, 23.16] and [5]).

If $N(\cdot)$ is regular, it possesses “boundary values” on the imaginary axis

$$(1) \quad N(i\eta) = \lim_{\xi \rightarrow 0+} N(\xi + i\eta) \in B(X), \quad \eta \in R,$$

where the limit exists in the strong operator topology (cf. [4, Theorem 17.9.1]).

It follows from (1) and the properties of $N(\cdot)$ that $\eta \rightarrow N(i\eta)$ is a strongly continuous *group* of operators which commutes with $N(\cdot)$ and satisfies the relation

$$(2) \quad N(\xi + i\eta) = N(\xi)N(i\eta) \quad (\xi \geq 0, \eta \in R).$$

Thus, a regular semigroup can be extended to form a strongly continuous semigroup in the *closed* half-plane \overline{C}^+ , holomorphic in C^+ . Conversely, if $N(\cdot)$ is a holomorphic semigroup in C^+ which admits a strongly continuous extension to \overline{C}^+ , then it follows from the Uniform Boundedness Theorem that $N(\cdot)$ is regular on the space

$$X_0 = \text{closure} \bigcup \{N(\zeta)X; \zeta \in C^+\},$$

which coincides with X when $N(\cdot)$ is of class (C_0) .

These remarks allow us to think of a regular semigroup as defined a priori in the *closed* half-plane.

It follows from the proof of Theorem 17.9.1 in [4] that $N(\cdot)$ is of exponential type $\leq \nu$ on \overline{C}^+ , i.e.

$$(3) \quad \|N(\zeta)\| \leq Ke^{\nu|\zeta|}, \quad \operatorname{Re} \zeta \geq 0,$$

for some constant $K > 0$. In particular,

$$1 = \|N(-i\eta)N(i\eta)\| \leq K^2 e^{2\nu|\eta|} \quad \text{for all } \eta \in \mathbf{R},$$

and so $\nu \geq 0$.

THEOREM 4.2. *Let $N(\cdot)$ be a regular semigroup of type $< \pi$. Then, for each ζ ($\operatorname{Re} \zeta \geq 0$), $N(\zeta)$ is one-to-one with dense range.*

Proof. 1. We show first that $Nx=0$ implies $x=0$. Consider the function $x(\zeta) = N(\zeta)x$ in $\operatorname{Re} \zeta \geq 0$ for x fixed such that $Nx=0$. It is strongly continuous in $\operatorname{Re} \zeta \geq 0$, holomorphic in $\operatorname{Re} \zeta > 0$, and of exponential type $\leq \nu < \pi$ in $\operatorname{Re} \zeta \geq 0$. Moreover, $x(k) = N^k x = 0$ for $k=1, 2, \dots$. By [4, Theorem 3.13.7], it follows that $x(\zeta) = 0$ for all ζ in $\operatorname{Re} \zeta \geq 0$; in particular ($\zeta=0$), $x=0$.

Suppose now that $N(\zeta_0)x=0$ for some $\zeta_0 = \xi_0 + i\eta_0$ ($\xi_0 \geq 0$) and $x \in X$. Then $N(\xi_0)x = N(-i\eta_0)N(\zeta_0)x = 0$. Let n be any integer $> \xi_0$. Then

$$N^n x = N(n)x = N(n - \xi_0)N(\xi_0)x = 0.$$

Since N is one-to-one (by the first part of the proof), it follows that $x=0$. Thus $N(\zeta)$ is one-to-one for each ζ in $\operatorname{Re} \zeta \geq 0$.

2. The conjugate group $\zeta \rightarrow N(\zeta)^*$ ($\zeta \in C^+$) is regular of type ν ($< \pi$), when restricted to the space

$$X_0^* = \text{closure } \bigcup_{\zeta \in C^+} N(\zeta)^* X^*.$$

Therefore, by the conclusion of 1, $N(\zeta)^*$ is one-to-one on X_0^* , for each fixed $\zeta \in \overline{C}^+$.

Suppose $N(\xi + i\eta)^* x^* = 0$ for some $x^* \in X^*$, $\xi > 0$ and $\eta \in \mathbf{R}$. Then $N(i\eta)^* N(\xi)^* x^* = 0$ and so $N(\xi)^* x^* = 0$ since $N(i\eta)$, and hence $N(i\eta)^*$, is nonsingular. For each $n=2, 3, \dots$, $N(\xi - \xi/n)^* N(\xi/n)^* x^* = 0$, and since $N(\xi/n)^* x^* \in X_0^*$ and $N(\xi - \xi/n)^*$ is one-to-one on X_0^* , it follows that $N(\xi/n)^* x^* = 0$. Since $N(\cdot)$ is of class (C_0) , we have, for each $x \in X$,

$$x^* x = \lim x^* N(\xi/n)x = \lim [N(\xi/n)^* x^*]x = 0.$$

Thus $x^* = 0$ and $N(\xi + i\eta)^*$ is one-to-one on X^* for $\xi > 0$; consequently, $N(\xi + i\eta)$ has dense range in X for $\xi > 0$ (the same is trivially true for $\xi=0$, since $N(i\eta)$ is nonsingular).

(The hypothesis $\nu < \pi$ is superfluous. The same proof, with trivial changes only, shows that the conclusion is valid for any holomorphic semigroup of class (C_0) in C^+ .)

REMARK 4.3. By Theorem 4.2, the inverse $N(\zeta)^{-1}$ exists as a densely defined closed operator with domain $N(\zeta)X$. If we let $N(-\zeta) = N(\zeta)^{-1}$, $\mathcal{D}_{-\zeta} = N(\zeta)X$ and $\mathcal{D}_{\zeta} = X$ ($\operatorname{Re} \zeta \geq 0$), we obtain a function $N(\cdot)$ defined on \mathbb{C} , whose values are closed operators $N(\zeta)$ with dense domain \mathcal{D}_{ζ} , which are bounded for $\operatorname{Re} \zeta \geq 0$. The semigroup property takes the form

$$N(\zeta_1 + \zeta_2) \supset N(\zeta_1)N(\zeta_2), \quad \zeta_1, \zeta_2 \in \mathbb{C},$$

and coincides with the usual identity if either $\operatorname{Re} \zeta_i \geq 0$ or $\operatorname{Re} \zeta_i \leq 0$ for both $i=1, 2$.

We shall think of $N(\cdot)$ as defined in the above sense for all $\zeta \in \mathbb{C}$. Note that $\mathcal{D}_{\zeta+i\eta} = \mathcal{D}_{\zeta}$ since $N(i\eta)$ is nonsingular ($\xi, \eta \in \mathbb{R}$).

THEOREM 4.4. Let $N(\cdot)$ be a regular semigroup of type $< \pi$. Then every operator which commutes with N commutes with $N(\zeta)$ for all $\zeta \in \mathbb{C}$.

Proof. Suppose $A \in B(X)$ commutes with N . For x fixed, the function $x(\zeta) = [A, N(\zeta)]x$ is strongly continuous in $\operatorname{Re} \zeta \geq 0$, holomorphic in $\operatorname{Re} \zeta > 0$, and of exponential type $< \pi$ in $\operatorname{Re} \zeta \geq 0$. Moreover, $x(k) = 0$ for $k=1, 2, \dots$. By [4, Theorem 3.13.7], $x(\zeta) = 0$ for all ζ in $\operatorname{Re} \zeta \geq 0$, and since x was arbitrary, the theorem is proved for $\operatorname{Re} \zeta \geq 0$.

Let $\zeta \in \mathbb{C}^+$ and $x \in \mathcal{D}_{-\zeta}$. Thus $x = N(\zeta)y$ ($y \in X$), $y = N(-\zeta)x$, and by the first part of the proof,

$$Ax = AN(\zeta)y = N(\zeta)Ay \in \mathcal{D}_{-\zeta}.$$

Hence

$$N(-\zeta)Ax = Ay = AN(-\zeta)x,$$

i.e. $N(-\zeta)A = AN(-\zeta)$ on $\mathcal{D}_{-\zeta}$. Q.E.D.

STANDING HYPOTHESIS 4.5. From now through the end of §6, $N(\cdot)$ is a regular semigroup of type $\nu < \pi$, and $A, M \in B(X)$ are such that

- (i) $A \neq 0$ commutes with M and N ; and
- (ii) $[N, M] = AN^2$.

We fix the notation $\mathcal{D}_{\zeta} = \text{domain } N(\zeta)$ (cf. Remark 4.3) and

$$(4) \quad T_{\zeta} = M + \zeta AN, \quad \zeta \in \mathbb{C}.$$

We now have the following generalization of [10, Lemma 1]:

THEOREM 4.6. For all $\zeta, \alpha \in \mathbb{C}$, \mathcal{D}_{ζ} is invariant under M and T_{α} , and the following equivalent identities are valid on \mathcal{D}_{ζ} :

- (a) $[N(\zeta), M] = \zeta AN(\zeta + 1)$;
- (b) $N(\zeta)M = T_{\zeta}N(\zeta)$;
- (c) $MN(\zeta) = N(\zeta)T_{-\zeta}$.

Proof. The three identities are equivalent because A commutes with $N(\zeta)$ by Theorem 4.4. Fix $x \in X$, and let

$$x(\zeta) = [N(\zeta), M]x - \zeta AN(\zeta + 1)x, \quad \operatorname{Re} \zeta \geq 0.$$

Fix $\varepsilon > 0$ such that $\nu + \varepsilon < \pi$. The vector-valued function $x(\cdot)$ is strongly continuous in $\operatorname{Re} \zeta \geq 0$, holomorphic in $\operatorname{Re} \zeta > 0$, and of exponential type $\leq \nu + \varepsilon < \pi$: by (3),

$$\|x(\zeta)\| \leq 2K\|M\| \|x\| e^{\nu|\zeta|} + K\|A\| \|N\| \|x\| |\zeta| e^{\nu|\zeta|} \leq K' e^{(\nu+\varepsilon)|\zeta|}.$$

By Lemma 2.1, $x(k)=0$ for $k=1, 2, \dots$. Therefore, by [4, Theorem 3.13.7], $x(\zeta)=0$ for all $\zeta \in \overline{C}^+$. Since x was arbitrary, this proves (a) (and hence (b) and (c)) for $\operatorname{Re} \zeta \geq 0$. By (c), $\mathcal{D}_{-\zeta}$ is invariant under M , and hence also under T_α by Theorem 4.4 ($\zeta \in C^+$, $\alpha \in C$). Let $x \in \mathcal{D}_{-\zeta}$, say $x = N(\zeta)y$ ($\zeta \in C^+$); then $y = N(-\zeta)x$ and by (c), for $\operatorname{Re} \zeta > 0$,

$$N(-\zeta)Mx = N(-\zeta)MN(\zeta)y = N(-\zeta)N(\zeta)T_{-\zeta}y = T_{-\zeta}y = T_{-\zeta}N(-\zeta)x.$$

This proves (b) (and hence all three identities) for $\operatorname{Re} \zeta < 0$.

COROLLARY 4.7. *For all $\zeta, \alpha \in C$,*

$$(5) \quad T_\zeta = N(\zeta - \alpha)T_\alpha N(\alpha - \zeta) \quad \text{on } \mathcal{D}_{\alpha - \zeta}.$$

In particular, T_ζ is similar to T_α if $\operatorname{Re} \zeta = \operatorname{Re} \alpha$, and the similarity is implemented by $N(i\eta)$ with $\eta = \operatorname{Im}(\alpha - \zeta)$.

Note that (5) tells that T_ζ is “unboundedly similar” to T_α for all $\zeta, \alpha \in C$.

Proof. By Theorem 4.6, we have on $\mathcal{D}_{\alpha - \zeta}$ (cf. Theorem 4.4)

$$T_\alpha N(\alpha - \zeta) = (T_{\alpha - \zeta} + \zeta AN)N(\alpha - \zeta) = N(\alpha - \zeta)(M + \zeta AN) = N(\alpha - \zeta)T_\zeta.$$

This implies the corollary.

REMARKS 1. By Lemma 3.1, AN is quasi-nilpotent. Let $\zeta \in C^+$ and fix an integer n such that $\operatorname{Re} n\zeta > 1$. Then, by Theorem 4.4, $[AN(\zeta)]^n = [A^{n-1}N(n\zeta - 1)]AN$ is quasi-nilpotent, and therefore $AN(\zeta)$ is quasi-nilpotent. For $\eta \in \mathbf{R}$, observe that the nonsingular operator $N(i\eta)$ is the strong operator limit of the quasi-nilpotent operators $N(\xi + i\eta)$ as $\xi \rightarrow 0+$ (for A nonsingular).

2. Suppose λ is an eigenvalue of T_α for some $\alpha \in C$. If $x (\neq 0)$ is a corresponding eigenvector, then for all $\zeta \in C$ with $\operatorname{Re} \zeta \geq \operatorname{Re} \alpha$, we have, by Corollary 4.7,

$$T_\zeta N(\zeta - \alpha)x = N(\zeta - \alpha)T_\alpha x = \lambda N(\zeta - \alpha)x.$$

By Theorem 4.2, $N(\zeta - \alpha)x \neq 0$, and therefore λ is an eigenvalue of T_ζ (with eigenvector $N(\zeta - \alpha)x$). Thus $\sigma_p(T_\alpha) \subset \sigma_p(T_\zeta)$ for $\operatorname{Re} \alpha \leq \operatorname{Re} \zeta$.

5. The C^k -classification of T_ζ .

HYPOTHESIS. 4.5.

LEMMA 5.1. *Let k be a nonnegative integer. Let $\zeta_j = \xi_j + i\eta_j$ ($j=1, 2$; $\xi_1 < \xi_2$; $\eta_j \in \mathbf{R}$) be such that T_{ζ_j} are real of class C^k .*

Then T_ζ is real of class C^k for all $\zeta = \xi + i\eta$ in the strip $\xi_1 \leq \xi \leq \xi_2$. Moreover $\sigma(T_\zeta) \subset \sigma(T_{\zeta_1}) \cup \sigma(T_{\zeta_2})$.

Proof. Constants will be denoted by K, K_1, K_2, \dots . Fix a polynomial φ , and consider the operator-valued entire function

$$\Phi(\zeta) = \exp(\nu\zeta^2)\varphi(T_\zeta) = \exp(\nu\zeta^2)N(i\eta)\varphi(T_\xi)N(-i\eta)$$

(cf. Corollary 4.7). We have

$$(1) \quad \begin{aligned} \|\Phi(\zeta)\| &\leq K^2 \exp(\nu(\xi^2 - \eta^2 + 2|\eta|))\|\varphi(T_\xi)\| \\ &\leq K^2 \exp(\nu(\xi^2 + 1))\|\varphi(T_\xi)\|. \end{aligned}$$

Thus Φ is bounded in the strip $\xi_1 \leq \xi \leq \xi_2$, and therefore, by the “three lines theorem” [2, Theorem VI.10.3],

$$\|\Phi(\xi + i\eta)\| \leq K_1 \max\{\|\varphi(T_{\xi_1})\|, \|\varphi(T_{\xi_2})\|\} \quad (\xi_1 \leq \xi \leq \xi_2),$$

i.e.

$$(2) \quad \|\varphi(T_{\xi+i\eta})\| \leq K_1 \exp(\nu(\eta^2 - \xi^2)) \max\{\|\varphi(T_{\xi_1})\|, \|\varphi(T_{\xi_2})\|\}$$

for all $\xi_1 \leq \xi \leq \xi_2$, $\eta \in \mathbf{R}$ and all polynomials φ . By Corollary 4.7, T_{ξ_j} is similar to T_{ζ_j} , and is therefore real of class C^k . Thus, there exists $K_2 > 0$ such that $\|\varphi(T_{\xi_j})\| \leq K_2 \|\varphi\|_k$ ($j=1, 2$) for all polynomials φ . Here $\|\varphi\|_k$ denotes the $C^k[a, b]$ -norm of φ , where $(a, b) \supset \sigma(T_{\xi_j})$ for $j=1, 2$. By (2),

$$\|\varphi(T_{\xi+i\eta})\| \leq K_3 \exp(\nu(\eta^2 - \xi^2)) \|\varphi\|_k \quad (\xi_1 \leq \xi \leq \xi_2, \eta \in \mathbf{R})$$

for all polynomials φ ; hence $T_{\xi+i\eta}$ is of class $C^k[a, b]$ in the strip $\xi_1 \leq \xi \leq \xi_2$. In particular, $\sigma(T_{\xi+i\eta}) \subset [a, b]$, by [9, Lemma 2.7], and $T_{\xi+i\eta}$ is real of class C^k . Let $\varphi \rightarrow \varphi(T_{\xi+i\eta})$ denote the $C^k[a, b]$ -operational calculus for $T_{\xi+i\eta}$; then (2) is valid for all $\varphi \in C^k[a, b]$. Suppose $\varphi \in C^k[a, b]$ is such that

$$[\text{support } \varphi] \cap [\sigma(T_{\xi_1}) \cup \sigma(T_{\xi_2})] = 0.$$

Then $\varphi(T_{\xi_j})=0$ (cf. [9, Lemma 2.2]) for $j=1, 2$, and therefore $\varphi(T_{\xi+i\eta})=0$ for $\xi_1 \leq \xi \leq \xi_2$, by (2). Using [9, Lemma 2.2] again, we conclude that

$$\sigma(T_{\xi+i\eta}) \subset \sigma(T_{\xi_1}) \cup \sigma(T_{\xi_2}) \quad (\xi_1 \leq \xi \leq \xi_2; \eta \in \mathbf{R}).$$

We now have the following generalization of [10, Theorem 6].

THEOREM 5.2. *Let r, s be nonnegative integers. Suppose M is real of class C^r . Then T_ζ is of class C^{r+s} and $\sigma(T_\zeta) = \sigma(M)$ for all ζ in the strip $|\operatorname{Re} \zeta| \leq s$.*

Proof. If $s=0$ (so that the strip reduces to the imaginary axis), T_ζ is similar to M in the “strip” (Corollary 4.7), and the conclusions of the theorem are trivial. Let then s be a positive integer. By Corollaries 3.9 and 3.10, $T_{\pm s}$ are real of class C^{r+s} and $\sigma(T_{\pm s}) = \sigma(M)$. Therefore, by Lemma 5.1, T_ζ is real of class C^{r+s} and $\sigma(T_\zeta) \subset \sigma(M)$ for all ζ in the strip $|\operatorname{Re} \zeta| \leq s$. Fix ζ in the strip, and let $M' = T_\zeta$. Then $M = M' - \zeta AN$, and the triple (A, M', N) satisfies the Standing Hypothesis 4.5. Moreover, we

proved above that M' is real of class C^{r+s} . Therefore, by the first part of the proof with M' replacing M ,

$$\sigma(M) = \sigma(M' - \zeta AN) \subset \sigma(M') = \sigma(T_\zeta).$$

Hence $\sigma(M) = \sigma(T_\zeta)$. Q.E.D.

COROLLARY 5.3. *Let r, s be nonnegative integers. Suppose $\|e^{itM}\| = O(|t|^r)$ as $|t| \rightarrow \infty$ (t real). Then $\sigma(T_\zeta) = \sigma(M)$ and T_ζ is (real) of class C^{r+s+2} for all ζ in the strip $|\operatorname{Re} \zeta| \leq s$.*

Proof. Apply [9, Lemma 2.11] and Theorem 5.2.

COROLLARY 5.4. *Let X be a Hilbert space, and suppose $\|e^{itM}\| = O(1)$ as $|t| \rightarrow \infty$. Let s be a nonnegative integer. Then T_ζ is real of class C^s and $\sigma(T_\zeta) = \sigma(M)$ for all ζ in the strip $|\operatorname{Re} \zeta| \leq s$.*

Proof. Apply [8, Theorem 5, p. 175] and Theorem 5.2.

The next theorem generalizes Theorems 8 and 9 in [10]. Under the hypothesis of Theorem 5.2, it gives explicitly the C^{r+s} -operational calculus for T_ζ ($\zeta \in \mathbb{C}$, $|\operatorname{Re} \zeta| \leq s$). Fix $a < b$ such that $(a, b) \supset \sigma(M)$, and let $\varphi \rightarrow \varphi(M)$ denote the $C^r[a, b]$ -operational calculus for M . By Corollary 3.10, the $C^{r+s}[a, b]$ -operational calculus for T_s is given by $\varphi \rightarrow \varphi(T_s)$, where

$$(3) \quad \varphi(T_s) = \sum_{j=0}^s \binom{s}{j} A^j N^j \varphi'(M), \quad \varphi \in C^{r+s}[a, b].$$

THEOREM 5.5. *Let r, s be nonnegative integers, and suppose M is real of class C^r . Then, for $|\operatorname{Re} \zeta| \leq s$ and $\varphi \in C^{r+s}[a, b]$, $D_{\zeta-s}$ ($= \operatorname{range} N(s-\zeta) = \operatorname{domain} N(\zeta-s)$) is invariant under $\varphi(T_s)$, and the $C^{r+s}[a, b]$ -operational calculus $\varphi \rightarrow \varphi(T_\zeta)$ for T_ζ is given by*

$$(4) \quad \varphi(T_\zeta) = N(\zeta-s)\varphi(T_s)N(s-\zeta), \quad \varphi \in C^{r+s}[a, b].$$

For $-s \leq \operatorname{Re} \zeta \leq 0$, D_ζ is invariant under $\varphi(M)$ and (4) simplifies to

$$(5) \quad \varphi(T_\zeta) = N(\zeta)\varphi(M)N(-\zeta), \quad \varphi \in C^{r+s}[a, b].$$

Proof. For $|\operatorname{Re} \zeta| \leq s$, $N(s-\zeta)$ is bounded, and therefore the relation $N(s-\zeta)\varphi(T_\zeta) = \varphi(T_s)N(s-\zeta)$, which is true for polynomials φ by Corollary 4.7, remains valid for all $\varphi \in C^{r+s}[a, b]$, since both T_ζ and T_s are of class $C^{r+s}[a, b]$. Therefore $D_{\zeta-s} = \operatorname{range} N(s-\zeta)$ is invariant under $\varphi(T_s)$ and (4) holds for all $\varphi \in C^{r+s}[a, b]$.

Suppose now that $-s \leq \operatorname{Re} \zeta \leq 0$. Since $N(-\zeta) \in B(X)$ and $T_{\pm\zeta}$ are of class $C^{r+s}[a, b]$, the relations

$$(6) \quad \varphi(M)N(-\zeta) = N(-\zeta)\varphi(T_\zeta)$$

and

$$(7) \quad \varphi(T_{-\zeta})N(-\zeta) = N(-\zeta)\varphi(M),$$

which are valid for polynomials φ by Theorem 4.6, remain valid for all $\varphi \in C^{r+s}[a, b]$. In particular, $D_\zeta = \text{range } N(-\zeta)$ is invariant under $\varphi(M)$ by (6) and ((7) with $\zeta = -s$)

$$\varphi(T_s)N(s) = N(s)\varphi(M).$$

Thus, by (4),

$$\begin{aligned}\varphi(T_\zeta) &= N(\zeta-s)\varphi(T_s)N(s)N(-\zeta) = N(\zeta-s)N(s)\varphi(M)N(-\zeta) \\ &= N(\zeta)\varphi(M)N(-\zeta),\end{aligned}$$

and the proof is complete.

Theorem 5.2 can be sharpened when $r=0$. In this case, we shall obtain a generalization of Theorem 1 in [11]. We first need however a generalization of Theorem 6 there.

NOTATION.

$$(8) \quad T_\zeta(t) = \exp(itT_\zeta) \quad (\zeta \in \mathbf{C}, t \in \mathbf{R}).$$

$$(9) \quad c = \|AN\|.$$

Since N has dense range (Theorem 4.2) and $A \neq 0$, we have $c > 0$.

LEMMA 5.6. *There exists a constant $H > 0$ such that*

$$\|T_{\zeta+i\eta}(t)\| \leq H \|e^{itM}\| (1+c|t|)^{|\zeta|} e^{2\nu|\eta|}$$

for all $\xi, \eta, t \in \mathbf{R}$.

Proof. By Corollary 4.7,

$$(10) \quad T_{\zeta+i\eta}(t) = N(i\eta)T_\zeta(t)N(-i\eta),$$

and therefore, by 4.1(3),

$$(11) \quad K^{-2}e^{-2\nu|\eta|}\|T_\zeta(t)\| \leq \|T_{\zeta+i\eta}(t)\| \leq K^2e^{2\nu|\eta|}\|T_\zeta(t)\|.$$

Fix $t \in \mathbf{R}$, and let

$$(12) \quad \Phi_i(\xi) = \exp(\nu\xi^2)T_\zeta(t).$$

By (11),

$$(13) \quad \|\Phi_i(\xi+i\eta)\| \leq K^2 \exp(\nu(\xi^2+1))\|T_\zeta(t)\|.$$

In particular, $\Phi_i(\xi+i\eta)$ is bounded in the strip $k-1 \leq \xi \leq k$ for each integer k . By Lemma 3.1 and (13),

$$(14) \quad \|\Phi_i(k+i\eta)\| \leq K^2 \exp(\nu(k^2+1))\|e^{itM}\|(1+c|t|)^{|k|}.$$

By the "three-lines theorem" [2, Theorem VI.10.3], it follows easily that

$$\|\Phi_i(\xi+i\eta)\| \leq K^2 \exp(\nu(\xi^2+5/4))\|e^{itM}\|(1+c|t|)^{|\xi|},$$

and therefore

$$\|T_\zeta(t)\| \leq \exp(-\nu\xi^2)\|\Phi_i(\xi)\| \leq K^2e^{5\nu/4}\|e^{itM}\|(1+c|t|)^{|\xi|}.$$

By (11), the proof is complete.

THEOREM 5.7. *Suppose $\|e^{itM}\| \leq B$ for all real t . Then there exist a constant H and a positive upper semicontinuous function $C(\xi)$ on \mathbf{R} such that*

$$C(\xi)e^{-2\nu|\eta|} \leq (1+c|t|)^{-|\xi|} \|T_{\xi+i\eta}(t)\| \leq He^{2\nu|\eta|}$$

for all $\xi, \eta, t \in \mathbf{R}$.

Proof. The right estimate follows from Lemma 5.6. We proceed to prove the left estimate. Note first that $\sigma(M)$ is real (cf. [8, p. 166]) and $1 = r(e^{itM}) \leq \|e^{itM}\|$. For $t, \xi \in \mathbf{R}$, let

$$C_t(\xi) = (1+c|t|)^{-|\xi|} \|T_t(\xi)\|, \quad \text{and} \quad C(\xi) = \inf \{C_t(\xi); t \in \mathbf{R}\}.$$

Clearly $C_t(\cdot)$ is continuous for each $t \in \mathbf{R}$, and therefore $C(\cdot)$ is upper semicontinuous. By (11), we must only prove that $C(\xi) > 0$ for all $\xi \in \mathbf{R}$. Suppose $C(\xi) = 0$ for some $\xi \in \mathbf{R}$. Then $\xi \neq 0$ since $C(0) = 1$. Fix an integer n such that $n|\xi| > 1$. There exists a sequence $\{t_k\} \subset \mathbf{R}$ such that $|t_k| \rightarrow \infty$ and $C_{t_k}(\xi) \rightarrow 0$ as $k \rightarrow \infty$. Fix $\varepsilon > 0$, and then k_0 such that

$$(15) \quad C_{t_k}(\xi) < \varepsilon^{|\xi|} \quad \text{for } k \geq k_0.$$

For $k \geq k_0$ fixed, consider the entire functions

$$F_k^\pm(\zeta) = (1+c|t_k|)^{\mp \zeta} \Phi_{t_k}(\zeta),$$

where Φ_t is defined by (12). By (13), these functions are bounded in each vertical strip $a \leq \operatorname{Re} \zeta \leq b$ ($a < b$ real), and

$$(16) \quad \|F_k^\pm(i\eta)\| \leq K^2 B e^\nu \quad (\text{cf. (14)}).$$

It follows easily from Theorem 3.2 that

$$\|T_{n\xi}(t)\| = \|e^{itM}[e^{-itM}T_t(t)]^n\| \leq B^{n+1} \|T_t(t)\|^n.$$

Therefore $C_t(n\xi) \leq B^{n+1} C_t(\xi)^n$, and by (15),

$$(17) \quad C_{t_k}(n\xi) \leq B^{n+1} \varepsilon^{n|\xi|}.$$

We determine the superscript of F_k as $(+)$ if $\xi > 0$ and $(-)$ if $\xi < 0$.

By (13) and (17),

$$\begin{aligned} \|F_k^\pm(n\xi + i\eta)\| &\leq K^2 \exp(\nu(n^2\xi^2 + 1)) C_{t_k}(n\xi) \\ &\leq K^2 B^{n+1} \exp(\nu(n^2\xi^2 + 1)) \varepsilon^{n|\xi|}. \end{aligned}$$

We now apply the "three-lines theorem" to F_k^+ (resp. F_k^-) in the strip $0 \leq \operatorname{Re} \zeta \leq n\xi$ when $\xi > 0$ (resp. $n\xi \leq \operatorname{Re} \zeta \leq 0$ when $\xi < 0$); we obtain (cf. (16))

$$\|F_k^\pm(\alpha + i\eta)\| \leq K^2 B^{n+1} \exp(\nu(n^2\xi^2 + 1)) \varepsilon^{|\alpha|} = K_1 \varepsilon^{|\alpha|}$$

in the respective strips.

Since $n|\xi| > 1$, this is true in particular for $\zeta = \alpha = 1$ (resp. -1). Thus

$$(1 + c|t_k|)^{-1} \|T_{\pm 1}(t_k)\| \leq K_1 \varepsilon \quad \text{for } k \geq k_0.$$

Thus

$$\lim_{k \rightarrow \infty} (1 + c|t_k|)^{-1} \|T_{\pm 1}(t_k)\| = 0.$$

By Lemma 3.1, however, $\|1 \pm it_k AN\| \leq B \|T_{\pm 1}(t_k)\|$. Therefore

$$\lim_{k \rightarrow \infty} (1 + c|t_k|)^{-1} \|1 \pm it_k AN\| = 0.$$

Since $|t_k| \rightarrow \infty$ and $c = \|AN\|$, this limit is trivially equal to 1, and we reached the wanted contradiction.

It is now very easy to prove the following generalization of Theorem 1 in [11].

THEOREM 5.8. *Let s be a nonnegative integer, and let M be real of class C . Then T_ζ is of class C^s if and only if $|\operatorname{Re} \zeta| \leq s$.*

Proof. By Theorem 5.2, T_ζ is real for all $\zeta \in \mathbb{C}$, and is of class C^s if $|\operatorname{Re} \zeta| \leq s$. Suppose now that T_ζ is of class C^s for some $\zeta = \xi + i\eta$ with $|\xi| > s$. Since T_ζ is real, it follows that, as $|t| \rightarrow \infty$, $\|T_{\xi + i\eta}(t)\| = O(|t|^s)$, hence $(1 + c|t|)^{-|\xi|} \|T_{\xi + i\eta}(t)\| = O(|t|^{s - |\xi|}) = o(1)$, contradicting Theorem 5.7 (since $\|e^{itM}\| \leq B$ is valid for M real of class C).

In Hilbert space, the conditions “ M real of class C ” and $\|e^{itM}\| = O(1)$ are equivalent (cf. [8, Theorem 5, p. 175]). Thus Corollary 5.4 has a full converse; formally (for X a Hilbert space)

COROLLARY 5.9. *Let s be a nonnegative integer, and suppose $\|e^{itM}\| = O(1)$. Then T_ζ is of class C^s if and only if $|\operatorname{Re} \zeta| \leq s$.*

6. Similarity and spectrality.

HYPOTHESIS. 4.5.

By Corollary 4.7, T_ζ is similar to T_α if $\operatorname{Re} \zeta = \operatorname{Re} \alpha$. As a consequence of Theorem 5.7, we obtain the following converse, which is a generalization of the deep part of Theorem 2 in [11]. The case $\operatorname{Re} \zeta = -\operatorname{Re} \alpha$ cannot be studied by these methods (in [11], this case is almost trivial, cf. §3). Besides the Standing Hypothesis 4.5, we assume that $\|e^{itM}\| = O(1)$, as in Theorem 5.7.

THEOREM 6.1. *T_ζ and T_α are not similar if $|\operatorname{Re} \zeta| \neq |\operatorname{Re} \alpha|$.*

Proof. By Corollary 4.7, it suffices to show that if ξ and λ are real and $|\xi| > |\lambda|$, then T_ξ is not similar to T_λ . Suppose Q is a nonsingular operator such that $T_\xi = Q^{-1}T_\lambda Q$. Then, by Theorem 5.7, we have

$$\begin{aligned} 0 < C(\xi) &\leq (1 + c|t|)^{-|\xi|} \|T_\xi(t)\| = (1 + c|t|)^{-|\xi|} \|Q^{-1}T_\lambda(t)Q\| \\ &\leq H \|Q\| \|Q^{-1}\| (1 + c|t|)^{|\lambda| - |\xi|} \rightarrow 0 \quad \text{as } |t| \rightarrow \infty, \end{aligned}$$

a contradiction.

The following generalization of Theorem 3 in [11] follows also from Theorem 5.7. The case $\operatorname{Re} \zeta = 0, 1, 2, \dots$ cannot be decided by this method.

THEOREM 6.2. *Suppose $\|e^{itM}\| = O(1)$. Then T_ζ is not spectral for $\operatorname{Re} \zeta \neq 0, 1, 2, \dots$*

Proof. By Corollary 4.7, it suffices to prove that T_ξ is not spectral for ξ real $\neq 0, 1, 2, \dots$. Suppose first that ξ is not an integer, and let n be the unique positive integer such that $n-1 < |\xi| < n$. Suppose T_ξ spectral, i.e., $T_\xi = S + Q$ with S a real scalar operator and Q a quasi-nilpotent operator commuting with S . By Theorem 5.7 (cf. [10, proof of Lemma 14(a)]), we have $Q^{n+1} = 0$, and thus

$$\sum_{k=0}^n (it)^k Q^k / k! = e^{itQ} = e^{-itS} T_\xi(t).$$

Since S is real and scalar, $\|e^{itS}\| \leq K$ for all $t \in \mathbf{R}$. Now, by Theorem 5.7, we have, as $|t| \rightarrow \infty$,

$$\begin{aligned} \|Q^n\| / (n! c^n) &= \lim (1 + c|t|)^{-n} \left\| \sum_{k=0}^n (it)^k Q^k / k! \right\| \\ &= \lim (1 + c|t|)^{-n} \|e^{-itS} T_\xi(t)\| \\ &\leq K \limsup (1 + c|t|)^{-n} \|T_\xi(t)\| \\ &\leq K_1 \limsup (1 + c|t|)^{|\xi| - n} = 0. \end{aligned}$$

Thus $Q^k = 0$, and therefore, as $|t| \rightarrow \infty$,

$$(1 + c|t|)^{-|\xi|} \|T_\xi(t)\| \leq K(1 + c|t|)^{-|\xi|} \sum_{k=0}^{n-1} |t|^k \|Q^k\| / k! \rightarrow 0$$

since $|\xi| > n-1$. This contradicts Theorem 5.7 and proves our theorem for $\operatorname{Re} \zeta$ not an integer. Consider now T_{-n} for a positive integer n . By Lemma 3.1,

$$T_{-n}(t) = e^{itM}(1 - itAN)^n.$$

Suppose T_{-n} is spectral; then $T_{-n} = S + Q$ and $Q^{n+1} = 0$ as before. Thus

$$(1) \quad (1 - itAN)^n = e^{-itM} T_{-n}(t) = e^{-itM} e^{itS} \sum_{k=0}^n (it)^k Q^k / k!.$$

Let $x \in \operatorname{range} Q$, say $x = Qy$ ($y \in X$). Then

$$|t|^{-n} \|(1 - itAN)^n x\| \leq K_2 |t|^{-n} \left\| \sum_{k=0}^{n-1} (it)^k Q^{k+1} y / k! \right\| \rightarrow 0$$

as $|t| \rightarrow \infty$. However the left side tends to $\|(AN)^n x\|$ as $|t| \rightarrow \infty$. Thus $(AN)^n x = 0$. Since A commutes with N and N is one-to-one, it follows that $A^n x = 0$. Thus $A^n Q = 0$. However A commutes with T_{-n} , and therefore with Q (cf. [1, Theorem 5]). Hence $QA^n = 0$, and by (1), $(1 - itAN)^n A^n = e^{-itM} e^{itS}$. Since the right side is a bounded function of t , we must have $NA^{n+1} = 0$, and since N is one-to-one, $A^{n+1} = 0$.

However A cannot be nilpotent, for then AN is nilpotent, and the left estimate of Theorem 5.7 cannot hold for all $\xi, \eta, t \in \mathbf{R}$ (take $\eta=0$, ξ an integer larger than the order of nilpotency of AN , and $t \rightarrow \infty$). This contradiction completes the proof.

COROLLARY 6.3. *If $\dim X < \infty$, there exists no triple A, M, N satisfying the hypothesis of Theorem 6.2.*

Proof. If $\dim X < \infty$, every operator on X is spectral.

COROLLARY 6.4. *Suppose X separable, $\|e^{itM}\| = O(1)$ and $\sigma_p(T_1)$ uncountable. Then T_ζ is not spectral for all ζ with $\operatorname{Re} \zeta \neq 0$.*

Proof. If T_ζ were spectral, it should be of finite type by Theorem 5.7 (cf. [10, proof of Lemma 14(a)]), and since X is separable, $\sigma_p(T_\zeta)$ should be countable [3, Theorem 1]. But $\sigma_p(T_\zeta) \supset \sigma_p(T_1)$ for $\operatorname{Re} \zeta \geq 1$, by the last remark of §4. Hence $\sigma_p(T_\zeta)$ is uncountable for $\operatorname{Re} \zeta \geq 1$, and therefore T_ζ is not spectral for $\operatorname{Re} \zeta \geq 1$. Taking this together with Theorem 6.2, we obtain the wanted conclusion.

COROLLARY 6.5. *Suppose X is separable, M is a real scalar operator, and $\sigma_p(T_1)$ is uncountable. Then T_ζ is spectral if and only if $\operatorname{Re} \zeta = 0$.*

Proof. Apply Corollaries 4.7 and 6.4. This last corollary generalizes Theorem 3 in [11].

7. Generalizations. We begin with an interesting consequence of Lemma 2.1.

THEOREM 7.1. *Let A be a Banach algebra with identity, and let $m, n \in A$ be such that $D_n^2 m = 0$ (i.e., $[m, n] = c$ commutes with n). Then, for every complex function g analytic in a neighborhood of $\sigma(n)$, $m + g'(n)c$ is similar to m . More precisely,*

$$(1) \quad m + g'(n)c = e^{-g(n)} m e^{g(n)}$$

(g can be A -valued, if its values commute with m and n).

Proof. Given g as in the hypothesis, take $f = e^g$ in Lemma 2.1. Then

$$m e^{g(n)} - e^{g(n)} m = D_m f(n) = f'(n) D_m n = e^{g(n)} g'(n) c;$$

multiplying both sides by $e^{-g(n)}$, we obtain (1).

COROLLARY 7.2. *Let A be a Banach algebra with identity. Let $m, n_1, \dots, n_k \in A$ be such that n_j and $[m, n_j] = c_j$ commute with n_i ($i, j = 1, \dots, k$). Let g_i be complex functions analytic in a neighborhood of $\sigma(n_i)$ ($i = 1, \dots, k$). Then $m + \sum_{i=1}^k g'_i(n_i) c_i$ is similar to m . More precisely,*

$$(2) \quad m + \sum_{i=1}^k g'_i(n_i) c_i = \exp \left(- \sum_{i=1}^k g_i(n_i) \right) m \exp \left(\sum_{i=1}^k g_i(n_i) \right).$$

Proof. Let $1 \leq p < k$. Then

$$\left[m + \sum_{i=1}^p g'_i(n_i) c_i, n_{p+1} \right] = c_{p+1}$$

commutes with n_{p+1} . Applying Theorem 7.1 with $g = g_{p+1}$, $n = n_{p+1}$ and m replaced by $m + \sum_{i=1}^p g'_i(n_i)c_i$, we obtain

$$m + \sum_{i=1}^{p+1} g'_i(n_i)c_i = \exp(-g_{p+1}(n_{p+1})) \left[m + \sum_{i=1}^p g'_i(n_i)c_i \right] \exp(g_{p+1}(n_{p+1})).$$

The result follows now by induction on p , extended up to k .

THEOREM 7.3. *Let A be a Banach algebra with identity. Let $m, n_1, n_2, \dots \in A$ be such that n_j and $[m, n_j] = c_j$ commute with n_i ($i, j = 1, 2, \dots$). Let g_i be complex functions, analytic in a neighborhood of $\sigma(n_i)$ ($i = 1, 2, \dots$), such that $\sum_{i=1}^{\infty} g_i(n_i)c_i$ converges in A , (to an element s). Then $m + \sum_{i=1}^{\infty} g'_i(n_i)c_i$ converges in A , and is similar to m . More precisely*

$$(3) \quad m + \sum_{i=1}^{\infty} g'_i(n_i)c_i = e^{-s} m e^s.$$

Proof. For each $k = 1, 2, \dots$, let $s_k = \sum_{i=1}^k g_i(n_i)$. Then $\exp(\pm s_k) \rightarrow \exp(\pm s)$, and therefore, by Corollary 7.2,

$$m + \sum_{i=1}^k g'_i(n_i)c_i = \exp(-s_k)m \exp(s_k) \rightarrow e^{-s} m e^s.$$

For example, if m and n_i are as in Theorem 7.3 and $g_i(\lambda) = z_i \lambda^{k_i+1}/(k_i+1)$ (k_i non-negative integers, z_i complex), then $m + \sum_{i=1}^{\infty} z_i n_i^{k_i+1} c_i$ converges and is similar to m if $\sum_{i=1}^{\infty} z_i n_i^{k_i+1}/(k_i+1)$ converges.

THEOREM 7.4. HYPOTHESIS 4.5. *For $i = 1, 2, \dots$, let $\alpha_i, \zeta_i \in \mathbb{C}$, $\operatorname{Re} \zeta_i \geq 1$, $\zeta_i \neq 1$. Suppose $\sum \alpha_i (\zeta_i - 1)^{-1} N(\zeta_i - 1)$ converges in $B(X)$. Then $M + \sum \alpha_i AN(\zeta_i)$ converges in $B(X)$ and is similar to M .*

Proof. In the Banach algebra $B(X)$, take $m = M$ and $n_i = N(\zeta_i - 1)$. Then, by Theorem 4.6,

$$c_i = [M, N(\zeta_i - 1)] = -(\zeta_i - 1)AN(\zeta_i).$$

In the preceding example, take $k_i = 0$ and $z_i = -\alpha_i(\zeta_i - 1)^{-1}$.

COROLLARY 7.5. HYPOTHESIS 4.5. *For $i = 1, 2, \dots$, let α_i, ζ_i be as in Theorem 7.4. Let*

$$T_\alpha = M + \alpha AN + \sum_{i=1}^{\infty} \alpha_i AN(\zeta_i), \quad \alpha \in \mathbb{C}.$$

Then the operators T_α satisfy Corollary 4.7 and Theorems 5.2, 5.5, 5.7, 5.8, 6.1 and 6.2.

Proof. Let $M' = M + \sum \alpha_i AN(\zeta_i)$. Since $[N, M'] = [N, M]$ and $M' \sim M$ by Theorem 7.4, the triple (A, M', N) satisfies Hypothesis 4.5, $\|e^{tM'}\| = O(1)$ if $\|e^{tM}\| = O(1)$ and M' is real of class C' if M has this property. We may therefore replace M by M' in the conclusions of all the theorems mentioned above.

Note that α_i could be operators, provided that they commute with A , M and N . Since $AN(\zeta)$ is quasi-nilpotent for $\operatorname{Re} \zeta > 0$ (cf. concluding remarks of §4), we obtain easily the following results. In 7.6–7.10, we assume Hypothesis 4.5.

COROLLARY 7.6. *Let $\zeta \in C^+$, and let n be the first integer such that $\operatorname{Re} n\zeta \geq 1$ ($n\zeta > 1$ if ζ is real). Let f be a complex function analytic at 0 such that $f^{(i)}(0) = 0$ for $0 \leq i \leq n-1$. Then $M + f(AN(\zeta)) \sim M$.*

COROLLARY 7.7. *Let $r, s \geq 0$ be integers, and suppose f is analytic at 0 with $f(0)$ real. If M is real of class C^r , then $M + f(AN)$ is of class C^{r+s} for $|\operatorname{Re} f'(0)| \leq s$, and $\sigma(M + f(AN)) = \sigma(M) + f(0)$.*

Moreover, if M is real of class C , then $M + f(AN)$ is of class C^s if and only if $|\operatorname{Re} f'(0)| \leq s$.

COROLLARY 7.8. *Suppose $\|e^{itM}\| = O(1)$. Then $M + f(AN)$ is not spectral for $\operatorname{Re} f'(0) \neq 0, 1, 2, \dots$ (f as before).*

COROLLARY 7.9. *Let f, g be analytic at 0. If $f(0) = g(0)$ and $\operatorname{Re} f'(0) = \operatorname{Re} g'(0)$, then*

$$M + f(AN) \sim M + g(AN).$$

Conversely, if $\|e^{itM}\| = O(1)$ and $M + f(AN) \sim M + g(AN)$, then

$$f(0) = g(0) \quad \text{and} \quad |\operatorname{Re} f'(0)| = |\operatorname{Re} g'(0)|.$$

COROLLARY 7.10. *Let f be analytic at 0 and $\|e^{itM}\| = O(1)$. Then $M + f(AN) \sim M$ if and only if $f(0) = \operatorname{Re} f'(0) = 0$.*

We turn now to generalizations in another direction. First, in the Banach algebra context (§3), we may replace the commutation relation $[n, m] = an^2$ by $D_n^2 m = 0$, and look for situations in which a “substitution” $n' = f(n)$ reduces the commutation relation $[m, n] = c$ to $[m, n'] = an'^2$. By Lemma 2.1, we must have (for f analytic in a neighborhood of $\sigma(n)$) $f'(n)c = af(n)^2$ (where a and f must be chosen). A special case of interest is when $c = \varphi(n)$, with φ analytic in a neighborhood of $\sigma(n)$. With $a = 1$, we then look for f such that $(f'\varphi - f^2)(n) = 0$. In particular, f may be taken to satisfy the differential equation $\varphi f' - f^2 = 0$ (in a neighborhood of $\sigma(n)$), provided that a solution analytic in a neighborhood of $\sigma(n)$ exists. The results of §3 extend then to the present situation, with n replaced by $n' = f(n)$. For example, the commutation relation

$$(ii)_k \quad [m, n] = n^k \quad (k \geq 2 \text{ fixed})$$

gives the differential equation $z^k f' - f^2 = 0$, which has the entire solution $f(z) = (k-1)z^{k-1}$. Thus the results of our paper are valid when (ii) is replaced everywhere by $(ii)_k$, provided n (or N) is replaced in the conclusions by $n' = (k-1)n^{k-1}$ (or $N' = (k-1)N^{k-1}$).

In the context of Hypothesis 4.5, we may replace the commutation relation (ii) by the more general relation (with $A = 1$ for simplicity):

$$(ii)_\zeta \quad [N, M] = N(\zeta) \quad (\zeta \text{ fixed, } \operatorname{Re} \zeta \geq 0).$$

A slight modification of the proof of Theorem 4.6 gives

$$(1) \quad [N(\alpha), M] = \alpha N(\alpha + \zeta - 1)$$

for $\operatorname{Re} \alpha \geq \max(0, 1 - \operatorname{Re} \zeta)$. If $\xi = \operatorname{Re} \zeta \leq 1$, we may take $\operatorname{Re} \alpha = 1 - \xi$ in (1). Thus

$$(2) \quad [N(1 - \xi + it), M] = (1 - \xi + it)N(i(\eta + t))$$

for all $t \in \mathbf{R}$. But this is impossible, since $N(i(\eta + t))$ is nonsingular, and (2) implies that it is quasi-nilpotent (cf. [12, Theorem 1.3.1]; take $t \neq 0$).

Thus (ii) _{ζ} is possible only for $\operatorname{Re} \zeta > 1$ (cf. Theorem 4.6(a)). In this case, choose $\alpha = \zeta - 1$ in (1). Then

$$[N(\zeta - 1), M] = (\zeta - 1)N(2(\zeta - 1)) = (\zeta - 1)N(\zeta - 1)^2$$

i.e., $N' = N(\zeta - 1)$ satisfies (ii) with $A = (\zeta - 1)I$ (of course, $(\zeta - 1)N(\zeta - 1)$ satisfies (ii) with $A = 1$, but trouble arises when we look for a corresponding semigroup). If ζ is real, the semigroup $N'(\alpha) = N(\alpha(\zeta - 1))$ satisfies $N'(1) = N'$ and Hypothesis 4.5 (with $A = (\zeta - 1)I$), except that its type may be larger than π . However, the requirement $\nu < \pi$ was needed only in order to obtain the conclusions of Theorems 4.2, 4.4 and 4.6; but the first two are valid for $N'(\cdot)$ because they are valid for $N(\cdot)$, and the third follows easily from (1).

We conclude that the results of §§5–7 are valid if, in Hypothesis 4.5, we replace the commutation relation (ii) by (ii) _{ζ} with $\zeta > 1$, provided A , N and $N(\cdot)$ are replaced respectively by $(\zeta - 1)I$, $N' = N(\zeta - 1)$ and $N'(\alpha) = N(\alpha(\zeta - 1))$.

We described the reduction of (ii) _{ζ} to (ii). There is of course a reverse process: if N' satisfies (ii) (with $A = (\zeta - 1)I$), then $N = N'(1/(\zeta - 1))$ ($\zeta > 1$ fixed) satisfies (ii) _{ζ} , and the corresponding semigroup is $N(\alpha) = N'(\alpha(\zeta - 1))$ —cf. Theorem 4.6(a). In particular, this provides examples of pairs $N(\cdot)$, M satisfying (ii) _{ζ} .

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